

MAXIMAL RANK MAPS BETWEEN RIEMANNIAN MANIFOLDS WITH BOUNDED GEOMETRY*

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1 Introduction

Rough isometries, in the sense of M. Kanai [10], provide equivalence relations between non-compact Riemannian manifolds. M. Kanai showed that when two spaces are roughly isometric they share properties such as volume growth rate and the validity of isoperimetric inequalities. He accomplished that via approximating a Riemannian manifold by a combinatorial structure, he calls a net. He proved that complete Riemannian manifolds, whose Ricci curvature are bounded from below, are roughly isometric to nets. We provide background in section 2.

Here we study mappings with maximal rank $\pi : M \rightarrow B$, between complete non-compact Riemannian manifolds M and N with bounded geometry. O'Neill [16] gives necessary and sufficient conditions for a Riemannian submersion $\pi : M \rightarrow B$ to be trivial, i.e., to differ only by an isometry of M from the simplest type of Riemannian submersions, the projection $p_B : F \times B \rightarrow B$ of a Riemannian product manifold $F \times B$ on one of its factors B (see **Theorem 3.2**). In section 3 we review O'Neill's results and describe the properties of long curves in B lifted to M .

In section 4 we define two new properties of maximal rank onto mappings $\pi : M \rightarrow B$: uniformly roughly isometric fibers [**Definition 4.1**] and horizontal lift control [**Definition 4.2**]. Then we prove that if M and B are complete Riemannian manifolds with bounded geometry, and if π satisfies these two properties with trivial holonomy, then M is roughly isometric to the product $F \times B$ of the base manifold B and a fixed fiber F of M [**Theorem 4.3**].

2 Rough Isometries, Nets and Bounded Geometry

In this section we introduce notation, give a few definitions according to M.Kanai [10] and O'Neill [16], and state some results without proofs, providing references whenever necessary.

Rough isometries, a concept first introduced by M. Kanai [10] give equivalence relations, which will be of our interest.

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Definition 2.1 Let (M, δ) and (N, d) be metric spaces. A map $\varphi : M \rightarrow N$, not necessarily continuous, is called a **rough isometry**, if it satisfies the following two axioms:

(RI.1) There exist constants $A \geq 1, C \geq 0$, satisfying,

$$\frac{1}{A}\delta(p_1, p_2) - C \leq d(\varphi(p_1), \varphi(p_2)) \leq A\delta(p_1, p_2) + C, \quad \forall p_1, p_2 \in M$$

(RI.2) The set $\text{Im}\varphi := \{q = \varphi(p), \forall p \in M\}$ is **full** in N , i.e.

$$\exists \varepsilon > 0 : N = B_\varepsilon(\text{Im}\varphi) = \{q \in N : d(q, \text{Im}\varphi) < \varepsilon\}$$

In this case we say that $\text{Im}\varphi$ is ε -**full** in N .

It is immediate to verify that if $\varphi : M \rightarrow N$ and $\psi : N \rightarrow M$ are rough isometries, then the composition $\varphi \circ \psi : N \rightarrow N$ is also a rough isometry.

A rough inverse of φ , which we will denote by $\varphi^- : N \rightarrow M$ is defined as follows: for each $q \in N$, choose $p \in M$ such that $d(\varphi(p), q) < \varepsilon$. Such a p exists because of axiom (RI.2). φ^- is a rough isometry such that both $\delta(\varphi^- \circ \varphi(p), p)$ and $d(\varphi \circ \varphi^-(q), q)$ are bounded in $p \in M$ and in $q \in N$, respectively.

To study geometric properties of manifolds, which are invariant under rough isometries, we next introduce what is called in [10], a **net**. A net is a discrete or combinatorial structure that provides approximations of Riemannian manifolds.

Definition 2.2 Let P be a countable set. A family $N = \{N(p) : p \in P\}$ is called a **net structure** of P if the following conditions hold for all $p, q \in P$:

(N.1) $N(p)$ is a finite subset of P

(N.2) $q \in N(p)$ iff $p \in N(q)$

Let M be a complete Riemannian manifold, and let d be the induced metric. A subset P of M is said to be ε -**separated** for $\varepsilon > 0$, if $d(p, q) \geq \varepsilon$ whenever p and q are distinct points of P , and an ε -separated set is called **maximal** if it is maximal with respect to the order relation of inclusion.

We have the following,

Proposition 2.3 If P is a countable maximal ε -separated set in a Riemannian manifold (M, d) , then P is ε -full in M , where $\varepsilon > 0$.

Proof. We want to show that,

$$d(x, P) < \varepsilon, \quad \forall x \in M$$

If $x \in P$, then $d(x, P) = 0 < \varepsilon$.

If $x \in M \setminus P$, by the maximality of P , there exists $\bar{p} \in P$ such that $d(x, \bar{p}) < \varepsilon$, and finally the definition of infimum implies that $d(x, P) := \inf_{p \in P} d(x, p) \leq d(x, \bar{p}) < \varepsilon$. \square

Let P be a maximal ε -separated subset of M . We define a **net structure** $N = \{N(p) : p \in P\}$ of P by $N(p) = \{q \in P : 0 < d(p, q) \leq 2\varepsilon\}$. A maximal ε -separated subset of a complete Riemannian manifold with the net structure described above will be called an **ε -net in M** .

For a point $p \in P$, each element of $N(p)$ is called a **neighbor** of p . A sequence $p = (p_0, \dots, p_l)$ of points in P is called a *path* from p_0 to p_l of length l if each p_k is a neighbor of p_{k-1} . A net P is said to be **connected** if any two points in P are joined by a path. For points p and q of a connected net P , $\delta(p, q)$ denotes the minimum of the lengths of paths from p to q . This δ satisfies the axioms of metric and it is called, according to [10], the **combinatorial metric** of P .

We observe that an ε -net in a complete Riemannian manifold is connected if the manifold is connected (see [10]).

In what follows, we introduce some notation (c.f. [10]) and we define a **bounded geometry** condition for manifolds.

Let (M, g) be a Riemannian manifold with Levi-Civita connection ∇ and curvature tensor R .

The **Ricci curvature tensor** of (M, g) , at each $x \in M$ is a symmetric bilinear form Ric defined by

$$\begin{aligned} Ric : T_x M \times T_x M &\longrightarrow \mathbb{R} \\ (\xi, \mu) &\longmapsto Ric(\xi, \mu) := \text{trace}(\zeta \mapsto R(\xi, \zeta)\mu) \end{aligned}$$

If M is complete, the **injectivity radius at $x \in M$** is given by

$$\iota_x(M) := \sup\{r > 0 : \exp_x|_{B(x, r)} \text{ is a diffeomorphism}\}$$

and $\iota(M) := \inf\{\iota_x(M) : x \in M\}$ is called the *injectivity radius* of M .

Definition 2.4 *Let M be a complete m -dimensional Riemannian manifold. We say that M has **bounded geometry** if it satisfies:*

(BG.R) *the Ricci curvature is bounded from below by $-(m-1)k_M^2$, where k_M is a positive constant;*

(BG.I) *the injectivity radius $\iota(M)$ is positive.*

We recall that a complete Riemannian manifold satisfying a bounded geometry condition has its geometry reflected by that of any net that approximates the manifold (see [10], Lemma2.5).

3 Long Curves and O'Neill Diffeomorphisms

Here we review background from O'Neill [16] and Abreu-Suzuki [2] concerning mappings of maximal rank.

Let M and B be Riemannian manifolds with dimensions m and n , respectively, where $m \geq n$. We will denote by $\pi : M \rightarrow B$ an onto mapping with maximal rank n , that is, π and each of its derivative maps π_* are surjective.

We start recalling the definitions of horizontal and vertical vectors, and of a Riemannian submersion, according to [16].

A tangent vector on M which is tangent to a fiber is called **vertical**, and if it is orthogonal to a fiber it is called **horizontal**. So, if a vector field on M is always tangent to fibers, we say that it is vertical, and if it is always orthogonal to fibers, we say that it is horizontal.

Definition 3.1 *A **Riemannian submersion** $\pi : M \rightarrow B$ is an onto mapping, such that, π has maximal rank, and π_* preserves lengths of horizontal vectors.*

Because for all $x \in M$ each derivative map $\pi_*(x)$ of π is surjective, we can define the projections \mathcal{H} and \mathcal{V} of the tangent space of M onto the subspaces of horizontal and vertical vectors, respectively, which will be denoted, respectively by $(VT)_x$ and $(HT)_x$ for each $x \in M$. In that case, we can decompose each tangent space to M into the direct orthogonal sum $T_x M = (VT)_x \oplus (HT)_x$.

Recall, O'Neill[16] proved,

Theorem 3.2 (O'Neill) *Let $\pi : M \rightarrow B$ be a submersion of a complete Riemannian manifold M . Then π is trivial if and only if the tensor T and the group G of the submersion both vanish.*

O'Neill defines the tensor T on M , which is the second fundamental form of all fibers, by $T_E F = \mathcal{H}\nabla_{\mathcal{V}E}(\mathcal{V}F) + \mathcal{V}\nabla_{\mathcal{V}E}(\mathcal{H}F)$ for arbitrary vector fields E and F , where ∇ is the covariant derivative of M . The group G of the submersion is the holonomy group of the connection $\Gamma(x \in M \mapsto \mathcal{H}(T_x M) = (HT)_x$), with reference to the base point $O \in B$.

The unique horizontal vector property, as stated in **Lemma 3.3**, follows from the maximality of the rank of the onto mapping π .

Lemma 3.3 *Let $b \in B$ be fixed. For any $w \in T_b B$ and $x \in M$ such that $\pi(x) = b$, there exists a unique horizontal vector $v \in T_x M$ which is π -related to w , i.e. satisfying $v \in (HT)_x$ and $(\pi_*)_x(v) = w$.*

In the following Lema, with additional control from below over the length of horizontal vectors, one has control from below over the distance in M .

Lemma 3.4 *Let M and B be connected and geodesically complete. For any $x, x' \in M$, let $\Gamma_{min} \subset M$ be a minimal geodesic joining x to x' , and let $\gamma_{min} \subset B$ be a minimal geodesic joining $\pi(x)$ to $\pi(x')$. Assume that for all $b \in B$ and for all $x \in F_b$ there exist constants $\alpha \geq 1$ and $\beta > 0$, both independent of b and x , satisfying*

$$\frac{1}{\alpha} \|w\|_B - \beta \leq \|v\|_M \quad (1)$$

for all $w \in T_b B$, where v is the unique horizontal lift of w through x that we assume satisfies $\|v\|_M \leq 1$, where $\| \cdot \|_M, \| \cdot \|_B$ denote the inner product on TM and TB , respectively.

Then, $d_M(x, x') = \ell(\Gamma_{min}) \geq \frac{1}{\alpha} \ell(\gamma_{min}) - \beta = \frac{1}{\alpha} d_B(\pi(x), \pi(x')) - \beta$

Proof. The proof of this Lemma is in [2].

Recall the definition of a lift of a curve.

Definition 3.5 *Let $\gamma : [t_1, t_2] \rightarrow B$ be a smooth embedded curve in B . A curve $\Gamma : [t_1, t_2] \rightarrow M$ satisfying $\pi \circ \Gamma = \gamma$ is called a **lift** of γ .*

*If in addition, Γ is horizontal, i.e., $\Gamma'(t) \in (HT)_{\Gamma(t)}, \forall t \in [t_1, t_2]$, where $\Gamma(t_1) = x_0 \in M$ with $\gamma(t_1) = \pi(x_0)$, the curve Γ is called a **horizontal lift** of γ through x_0 . Recall that the horizontal lift of a curve in B , through a point $x_0 \in M$ is unique.*

We now define long curves [2].

Definition 3.6 Let $\beta > 0$ be any fixed constant. A smooth embedded curve $\gamma : [t_1, t_2] \rightarrow B$ is said to be a β -long curve if $\inf_{t_1 \leq t \leq t_2} \|\gamma'(t)\| \geq \beta$. In that case, $\ell(\gamma) \geq \int_{t_1}^{t_2} \|\gamma'(t)\| dt \geq \beta(t_2 - t_1)$. We say that a curve γ is simply a long curve if it is a β -long curve for some constant $\beta > 0$.

Let $\gamma : [t_1, t_2] \rightarrow B$ denote a smooth embedded curve and let $\Gamma : [t_1, t_2] \rightarrow M$ denote a lift of γ .

In the next two Propositions, proven in [2], under control from above (or below) on the derivative of the maximal rank mapping π , we have control from below (or above) over the length of any lift of a curve. In **Proposition 3.7** any lift Γ in M of a long curve γ in B cannot be short, and in **Proposition 3.8** the length of a lift Γ of a long curve γ is bounded above by the length of γ .

The Riemannian norms in TM and TB will be denoted by $\|\cdot\|_M$ and $\|\cdot\|_B$, respectively.

Proposition 3.7 Let $\alpha \geq 1$ and $\beta > 0$ be constants satisfying,

$$\|(\pi_*)_x v\|_B \leq \alpha \|v\|_M + \beta \quad (2)$$

for all $x \in M$, for all $v \in T_x M$ where $\|v\|_M \leq 1$.

If γ is any smooth β -long curve in B , then,

$$\ell(\Gamma) \geq \frac{1}{\alpha} [\ell(\gamma) - \beta(t_2 - t_1)] > 0$$

where $\ell(\Gamma)$ and $\ell(\gamma)$ denote the lengths of the curves Γ and γ , respectively.

Proposition 3.8 Let Γ be a lift of γ , and assume for horizontal^(†) vectors $v \in TM$ only, that there is a universal constant $\alpha \geq 1$ satisfying,

$$\|(\pi_*)_x v\|_B \geq \frac{1}{\alpha} \|v\|_M - \beta \quad (3)$$

for all $x \in M$, for all $v \in T_x M \setminus (VT)_x = (HT)_x = [\ker(\pi_*)_x]^\perp$.

For a β -long curve γ , we have,

$$\ell(\Gamma) \leq \alpha [\ell(\gamma) + \beta(t_2 - t_1)]$$

where $\ell(\Gamma)$ and $\ell(\gamma)$ denote the lengths of the curves Γ and γ , respectively.

Next, for onto smooth mappings with maximal rank between complete Riemannian manifolds, we recall the definition of special diffeomorphisms between any two fibers, we call **O’Neill diffeomorphisms**, a useful tool that will feature in many of our proofs.

Firstly, we give the definition, and in the five propositions that follow we state several of their properties (see [1] Theorem 4.12).

Definition 3.9 Let $\pi : M \rightarrow B$ be an onto smooth map with maximal rank, where M, B are complete and B is connected. Let b_1, b_2 be distinct elements of B and let $\gamma : [t_1, t_2] \rightarrow B$ be a piecewise smooth embedded curve parametrized proportionally to arclength, where $\gamma(t_1) = b_1, \gamma(t_2) = b_2$.

If we denote by $\pi^{-1}(b_1) = F_{b_1}$ and $\pi^{-1}(b_2) = F_{b_2}$ their corresponding fibers, we thus define the map $\varphi_{(\gamma)} : F_{b_1} \rightarrow F_{b_2}$, we refer to as **O’Neill diffeomorphism**, by the following rule: Given $x \in F_{b_1}$, let Γ_x be the unique horizontal lift of γ through x , and set $\varphi_{(\gamma)}(x) := \Gamma_x(t_2) \in F_{b_2}$. (see Fig. 1)

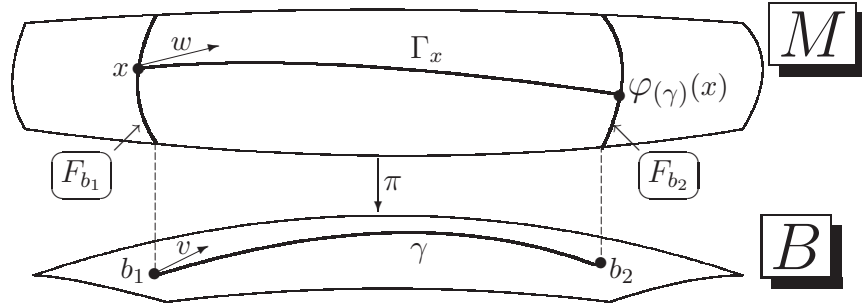
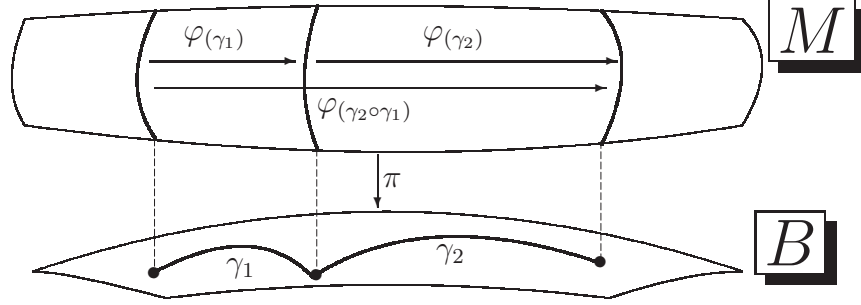


Figure 1: The map $\varphi_{(\gamma)}$.

O’Neill [16] noted that his diffeomorphisms have the following five properties (detailed proofs are available in author’s thesis [1]):

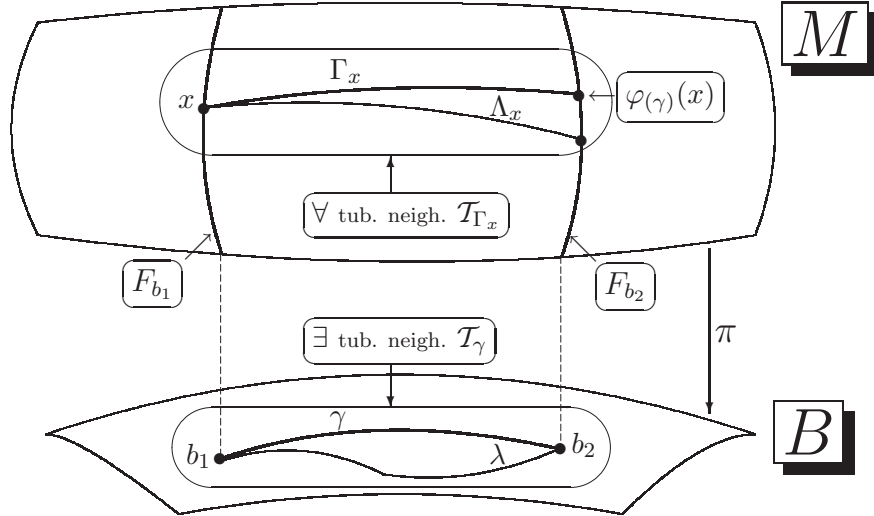
Proposition 3.10 $\varphi_{(\gamma)} : F_{b_1} \rightarrow F_{b_2}$ is well-defined.

Proposition 3.11 Let $\gamma_1 : [t_1, t_2] \rightarrow B$ and $\gamma_2 : [t_2, t_3] \rightarrow B$ be smooth embedded curves parametrized proportionally to arclength, such that $\gamma_1(t_2) = \gamma_2(t_2)$. Define $\gamma_3 : [t_1, t_3] \rightarrow B$ the composition of γ_1 and γ_2 , denoted by $\gamma_3 = \gamma_2 \circ \gamma_1$, as follows: $t \in [t_1, t_3] \mapsto \gamma_3(t) := \begin{cases} \gamma_1(t), & \text{if } t_1 \leq t \leq t_2 \\ \gamma_2(t), & \text{if } t_2 \leq t \leq t_3 \end{cases}$. Then $\varphi_{(\gamma_3)} : F_{\gamma_1(t_1)} \rightarrow F_{\gamma_2(t_3)}$ satisfies $\varphi_{(\gamma_3)} = \varphi_{(\gamma_2)} \circ \varphi_{(\gamma_1)}$. (see Fig. 2)

Figure 2: Property $\varphi(\gamma_2 \circ \gamma_1) = \varphi(\gamma_2) \circ \varphi(\gamma_1)$.

Proposition 3.12 $\varphi_{(\gamma)} : F_{b_1} \longrightarrow F_{b_2}$ is a diffeomorphism.

Proposition 3.13 $\varphi_{(\gamma)}$ depends continuously on γ , i.e., for sufficiently small displacements of γ within a tubular neighborhood, keeping the endpoints fixed, their horizontal lifts through x lie entirely within any given small tubular neighborhood of Γ_x . Furthermore, since horizontal lifts are, by definition, curves tangent to $(HT) \subset TM$, this fact forces the endpoints of horizontal lifts through x to belong to arbitrary balls around $\varphi_{(\gamma)}(x) = \Gamma_x(t_2)$, as long as those horizontal lifts through x lie entirely within a sufficiently small tubular neighborhood of Γ_x . (see Fig. 3)

Figure 3: $\varphi_{(\gamma)}$ depends continuously on γ .

Proposition 3.14 *Let $b \in B$ be a fixed base point and $\gamma_b : [t_1, t_2] \rightarrow B$ be a piecewise smooth embedded geodesic loop parametrized proportionally to arclength, where $\gamma_b(t_1) = \gamma_b(t_2) =: b$. The properties of $\varphi_{(\gamma_b)} : F_b \rightarrow F_b$, imply that the set of mappings $G_b := \{\varphi_{(\gamma_b)} : F_b \rightarrow F_b, \forall \gamma_b\}$, defines a group of diffeomorphisms of the fiber F_b , called the **holonomy** group of the assignment $x \in M \mapsto (HT)_x \subset T_x M$, with reference to the point $b \in B$. (see Fig. 4)*

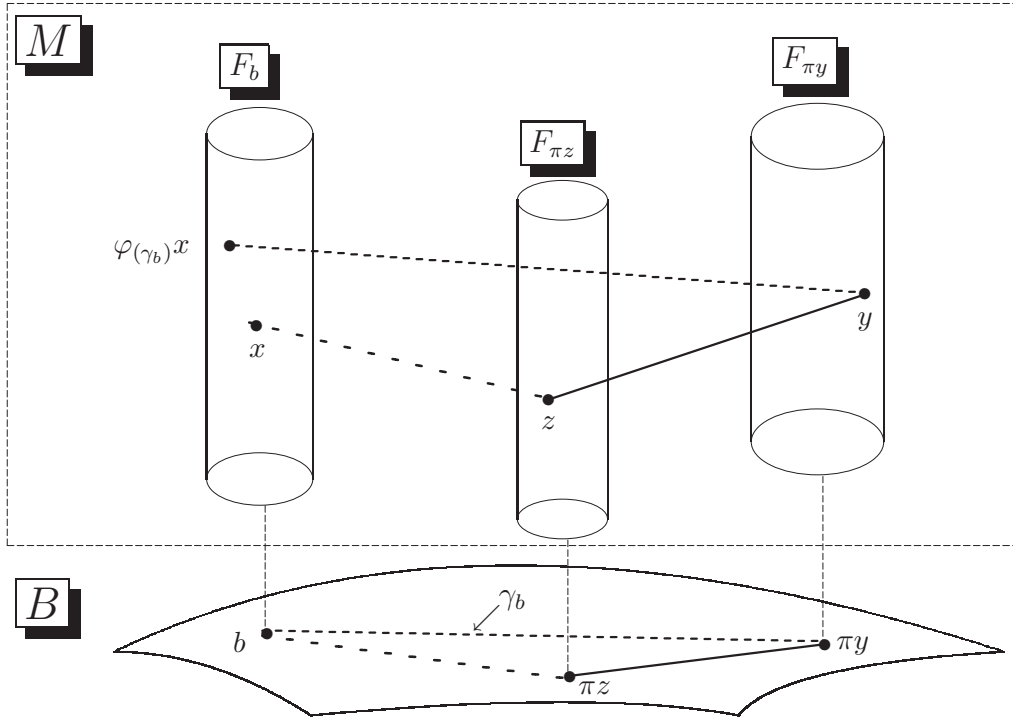


Figure 4: Holonomy Group G_b .

4 The Main Theorem

We begin this section with definitions of two new properties of maximal rank onto mappings: uniformly roughly isometric fibers [**Definition 4.1**] and horizontal lift control [**Definition 4.2**].

Let $\pi : M \rightarrow B$ be an onto smooth map with maximal rank between complete Riemannian manifolds M and B with dimensions m and n , re-

spectively.

Consider $b_0 \in B$ a fixed base point.

For every $b_1, b_2 \in B$ we will denote by $\gamma_{[b_1 b_2]}$ a broken geodesic in B joining b_1 to b_2 . In particular $\gamma_{b_1} := \gamma_{[b_1 b_1]}$ will denote a broken geodesic loop at b_1 . Let $\varphi_{(\gamma_{[b_1 b_2]})} : F_{b_1} \rightarrow F_{b_2}$ be the corresponding O'Neill diffeomorphism to $\gamma_{[b_1 b_2]}$, as in Definition 3.9.

According to O'Neill [16] an onto maximal rank map $\pi : M \rightarrow B$ has **trivial holonomy** with reference to the point b_0 , if for any broken geodesic loop γ_{b_0} , the corresponding O'Neill diffeomorphism $\varphi_{(\gamma_{b_0})} : F_{b_0} \rightarrow F_{b_0}$ is the identity map on F_{b_0} (see Fig. 5).

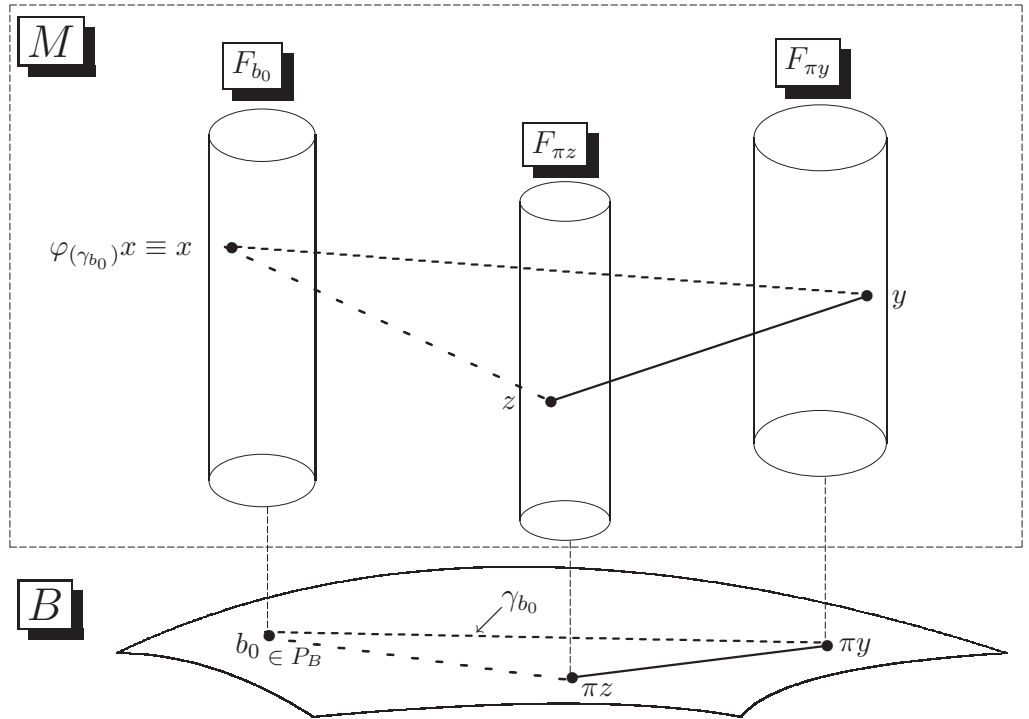


Figure 5: trivial holonomy with reference to the point b_0 .

Definition 4.1 An onto maximal rank map $\pi : M \rightarrow B$ has **uniformly roughly isometric fibers (RIF)** if for all $b \in B$ there exist constants $A > 1$ and $C > 0$, both independent of b , such that,

$$\frac{1}{A} d_M(x, x') - C \leq d_M(\varphi_{(\gamma_{[b, b_0]})}(x), \varphi_{(\gamma_{[b, b_0]})}(x')) \leq A d_M(x, x') + C$$

for all $x, x' \in F_b$, where d_M denotes the Riemannian metric on M .

In this case, since $\varphi_{(\gamma[b, b_0])}$ is onto, it follows that $\varphi_{(\gamma[b, b_0])} : F_b \longrightarrow F_{b_0}$ is a rough isometry for each $b \in B$, and therefore the fibers are uniformly roughly isometric (see Fig. 6).

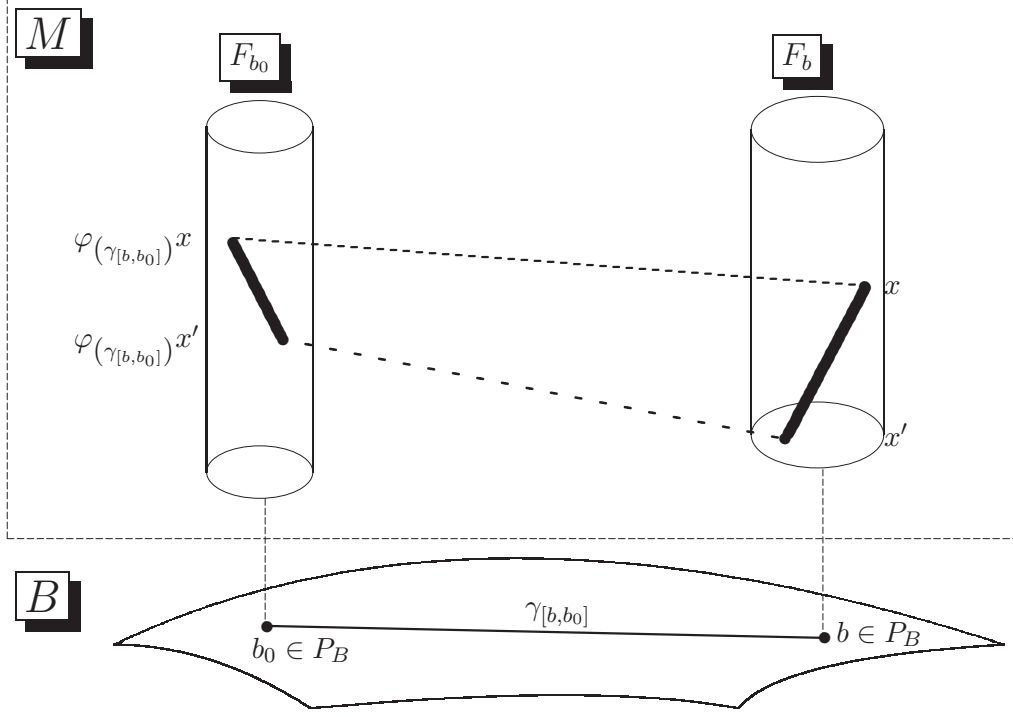


Figure 6: **(RIF)**.

Definition 4.2 An onto maximal rank map $\pi : M \rightarrow B$ has **horizontal lift control (HLC)** if for all $b \in B$ and for all $x \in F_b$ there exist constants $\alpha \geq 1$ and $\beta > 0$, both independent of b and x , such that

$$\frac{1}{\alpha} \|w\|_B - \beta \leq \|v\|_M \leq \alpha \|w\|_B + \beta$$

for all $w \in T_b B$, where v is the unique horizontal lift of w through x satisfying $\|v\|_M \leq 1$, and $\|\cdot\|_M, \|\cdot\|_B$ denote the inner product on TM and TB , respectively (see Fig. 7).

Finally, we state and prove the main result. **Theorem 4.3** was motivated by O'Neill's [16] question adapted for Mappings with Maximal Rank.

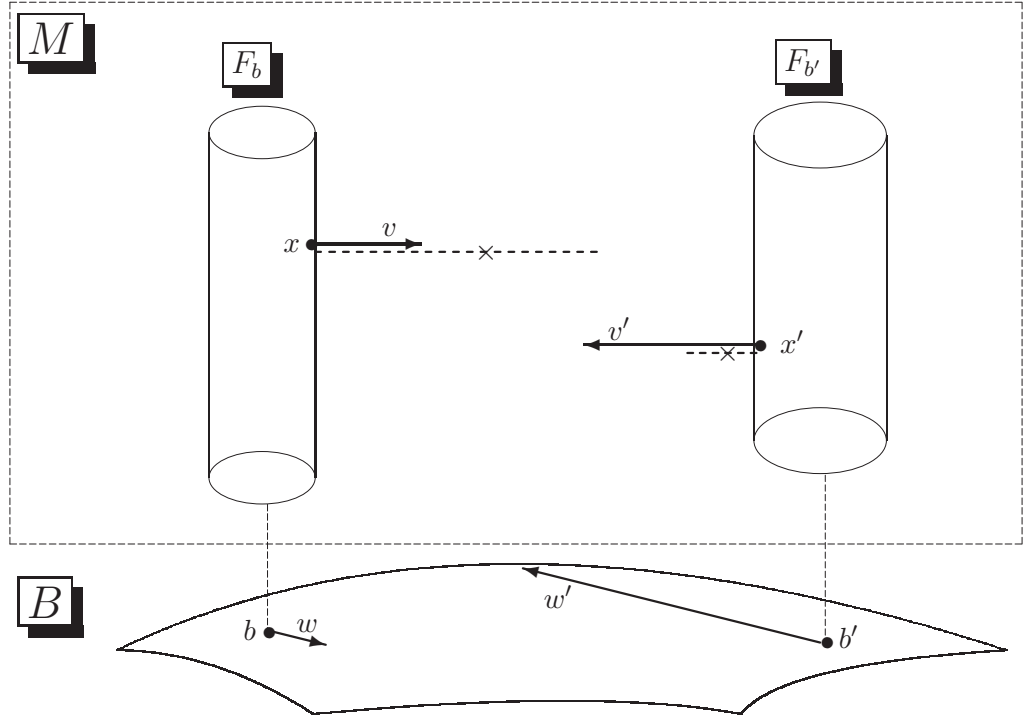


Figure 7: **(HLC)**, where w is short and w' is long.

Theorem 4.3 *Let M and B be complete Riemannian manifolds, with bounded geometry and dimensions m and n , respectively. Let $\pi : M \rightarrow B$ be an onto smooth maximal rank map, and let $b_0 \in B$ be a fixed base point. Assume that π has trivial holonomy, uniformly roughly isometric fibers **(RIF)**, and horizontal lift control **(HLC)**. Then, M is roughly isometric to the product $F_{b_0} \times B$.*

In order to prove **Theorem 4.3** we will need the following technical Lemma and Proposition.

Lemma 4.4 *Let $A > 1$, $C > 0$, $\alpha \geq 1$, $\beta > 0$, be given constants.*

For any positive real numbers $\epsilon_0 > 0$, $\epsilon_B > 0$ satisfying,

$$\epsilon_0 > C \cdot (1 + A^2) \quad \text{and} \quad \epsilon_B > \left(\frac{\epsilon_0 - C}{A} + \beta \right) \cdot \alpha \quad (4)$$

the following hold:

$$\left(\frac{\epsilon_0 - C}{A} \right) > 0 \quad (5)$$

$$\frac{\epsilon_0 - C}{A} < \epsilon_0 \quad (6)$$

$$\frac{\epsilon_0 - C}{A^2} - C > 0 \quad (7)$$

$$\frac{\epsilon_0 - C}{A} > \frac{\epsilon_0 - C}{A^2} - C \quad (8)$$

$$\frac{1}{\alpha} \epsilon_B - \beta > 0 \quad (9)$$

$$\frac{\epsilon_0 - C}{A} < (\epsilon_0 + C) \cdot A \quad (10)$$

$$2 \left(\frac{\epsilon_0 - C}{A} \right) < (2\epsilon_0 + C) \cdot A \quad (11)$$

$$2\epsilon_B > 2 \left(\frac{\epsilon_0 - C}{A} + \beta \right) \alpha \quad (12)$$

Proof.

(5): By the first inequality in (4),

$$\epsilon_0 > \underbrace{C(1 + A^2)}_{>1} > C \Rightarrow \left(\frac{\epsilon_0 - C}{A} \right) > 0$$

(6): Since $A > 1$,

$$\begin{aligned} \epsilon_0 \underbrace{(1 - A)}_{<0} < 0 < C &\Rightarrow \frac{\epsilon_0}{A}(1 - A) < \frac{C}{A} \Rightarrow \\ &\Rightarrow \frac{\epsilon_0}{A} - \epsilon_0 - \frac{C}{A} < 0 \Rightarrow \frac{\epsilon_0}{A} - \frac{C}{A} < \epsilon_0 \Rightarrow \\ &\Rightarrow \frac{\epsilon_0 - C}{A} < \epsilon_0 \end{aligned}$$

(7): By the first inequality in (4),

$$\epsilon_0 > C(1 + A^2) \Rightarrow \epsilon_0 - C - CA^2 > 0 \Rightarrow \frac{\epsilon_0 - C}{A^2} - C > 0$$

(8): Since $A > 1$,

$$\begin{aligned} \epsilon_0 \underbrace{(1 - A)}_{<0} < 0 < C + CA \underbrace{(A - 1)}_{>0} &\Rightarrow \\ \Rightarrow \epsilon_0(1 - A) < C + CA^2 - CA &\Rightarrow \epsilon_0 - C - CA^2 < \epsilon_0 A - CA \Rightarrow \\ \Rightarrow \frac{\epsilon_0 - C}{A^2} - C < \frac{\epsilon_0 - C}{A} \end{aligned}$$

(9): By the second inequality in (4),

$$\epsilon_B > \left(\frac{\epsilon_0 - C}{A} + \beta \right) \alpha = \underbrace{\left(\frac{\epsilon_0 - C}{A} \right) \alpha}_{>0} + \beta \alpha > \beta \alpha \Rightarrow$$

$$\stackrel{\div \alpha}{\Rightarrow} \frac{1}{\alpha} \epsilon_B - \beta > 0$$

(10): $A > 1, C > 0$ and $\epsilon_0 > 0$ imply that,

$$\epsilon_0(1 - A)(1 + A) < 0 \Rightarrow$$

$$\Rightarrow \epsilon_0(1 - A)(1 + A) < 0 < C(1 + A^2) \Rightarrow \epsilon_0(1 - A^2) < C(1 + A^2) \Rightarrow$$

$$\Rightarrow \epsilon_0 - \epsilon_0 A^2 < C + C A^2 \Rightarrow \epsilon_0 - C < \epsilon_0 A^2 + C A^2 \Rightarrow$$

$$\Rightarrow \frac{\epsilon_0 - C}{A} < (\epsilon_0 + C)A$$

(11): From $A > 1, C > 0$ and $\epsilon_0 > 0$ we have,

$$\begin{cases} 2\epsilon_0(1 - A)(1 + A) < 0 \\ 0 < C(2 + A^2) \end{cases} \Rightarrow 2\epsilon_0(1 - A)(1 + A) < 0 < 2C + C A^2 \Rightarrow$$

$$\Rightarrow 2\epsilon_0(1 - A^2) < 2C + C A^2 \Rightarrow 2\epsilon_0 - 2\epsilon_0 A^2 < 2C + C A^2 \Rightarrow$$

$$\Rightarrow 2\epsilon_0 - 2C < 2\epsilon_0 A^2 + C A^2 \Rightarrow$$

$$\Rightarrow 2 \left(\frac{\epsilon_0 - C}{A} \right) < (2\epsilon_0 + C)A$$

(12): By the second inequality in (4),

$$\begin{aligned} \epsilon_B &> \left(\frac{\epsilon_0 - C}{A} \right) \alpha + \beta \alpha > \left(\frac{\epsilon_0 - C}{A} \right) \alpha + \frac{\beta \alpha}{2} \Rightarrow \\ \stackrel{\times 2}{\Rightarrow} 2\epsilon_B &> 2 \left(\frac{\epsilon_0 - C}{A} \right) \alpha + \beta \alpha \Rightarrow \\ \Rightarrow 2\epsilon_B &> \left[2 \left(\frac{\epsilon_0 - C}{A} \right) + \beta \right] \alpha \end{aligned}$$

□

Proposition 4.5 *Suppose that M and B are complete m -dimensional and n -dimensional Riemannian manifolds, respectively, both with bounded geometry. Let $\pi : M \rightarrow B$ be an onto smooth map with maximal rank, and $b_0 \in B$ be fixed.*

Assume that $\{\phi_b : F_b \rightarrow F_{b_0}\}_{b \in B}$ is a family of bijective rough isometries satisfying,

$$\begin{aligned} \forall b \in B, \exists A > 1, \exists C > 0 : \\ \frac{1}{A}d_M(x, x') - C \leq d_M(\phi_b(x), \phi_b(x')) \leq Ad_M(x, x') + C, \quad (13) \\ \forall x, x' \in F_b \end{aligned}$$

where, A and C are universal constants independent of b .

Then, the following hold:

- If P_0 is an ϵ_0 -separated set and ϵ_0 -full in F_{b_0} , where we assume that $\epsilon_0 > C$, then the set

$$P_b := \phi_b^{-1}(P_0)$$

is an $\hat{\epsilon}$ -separated set and $\tilde{\epsilon}$ -full in F_b , where $\hat{\epsilon} := \left(\frac{\epsilon_0 - C}{A}\right) > 0$ and $\tilde{\epsilon} := (\epsilon_0 + C)A > 0$.

- For all $b \in B$ the corresponding nets P_b are uniformly roughly isometric to P_0 with respect to the combinatorial metric δ .

Proof. From Lemma 2.5 [10], a complete Riemannian manifold with bounded geometry is roughly isometric to each of its nets.

This implies that for each $b \in B$,

$$(P_b, \delta) \xrightarrow{\mathbf{R.I.}} (F_b, d_M) \Rightarrow$$

$$\frac{1}{2\hat{\epsilon}}d_M(p_1, p_2) \leq \delta(p_1, p_2) \leq \tilde{a}d_M(p_1, p_2) + \tilde{c}, \quad \forall p_1, p_2 \in P_b \quad (14)$$

where $\tilde{a} := \tilde{a}(m, k_M, \tilde{\epsilon}) > 1$, $\tilde{c} := \tilde{c}(m, k_M, \tilde{\epsilon}) > 0$, and (F_b, d_M) indicates that on each fiber F_b we will use the induced Riemannian metric from M .

Also, by [10] (Lemma 5) we have,

$$(P_0, \delta_0) \xrightarrow{\mathbf{R.I.}} (F_0, d_M) \Rightarrow$$

$$\frac{1}{2\epsilon_0}d_M(p_3, p_4) \leq \delta_0(p_3, p_4) \leq \tilde{a}_0d_M(p_3, p_4) + \tilde{c}_0 \Rightarrow$$

$$\frac{1}{\tilde{a}_0} \delta_0(p_3, p_4) - \frac{\tilde{c}_0}{\tilde{a}_0} \leq d_M(p_3, p_4) \leq 2\epsilon_0 \delta_0(p_3, p_4), \quad \forall p_3, p_4 \in P_0 \quad (15)$$

where $\tilde{a}_0 := \tilde{a}_0(m, k_M, \epsilon_0) > 1$, $\tilde{c}_0 := \tilde{c}_0(m, k_M, \epsilon_0) > 0$, and (F_0, d_M) indicates that on the fiber F_0 the induced Riemannian metric from M is used.

From (??), for all $p_1, p_2 \in F_b$,

$$(F_b, d_M) \xrightarrow{\mathbf{R.I.}} (F_0, d_M) \Rightarrow$$

$$\frac{1}{A} d_M(\phi_b(p_1), \phi_b(p_2)) - \frac{C}{A} \leq d_M(p_1, p_2) \leq A d_M(\phi_b(p_1), \phi_b(p_2)) + AC \quad (16)$$

Next, we observe the following diagram for $\iota = 1, 2$,

$$\begin{array}{ccccccc} P_b & & F_b & & F_0 & & P_0 \\ p_\iota & \hookrightarrow & p_\iota & \xrightarrow{\phi_b} & \phi_b(p_\iota) & \hookrightarrow & \phi_b(p_\iota) \end{array}$$

where, $p_\iota \in P_b := \phi_b^{-1} P_0 \Rightarrow \phi_b(p_\iota) \in P_0$.

We claim that,

$$(P_b, \delta) \xrightarrow{\text{unif. R.I.}} (P_0, \delta_0)$$

Indeed, let $p_1, p_2 \in P_b$.

By (14), (16) and (15), we may write,

$$\begin{aligned} \xRightarrow{(14)} \quad & \frac{1}{2\hat{\epsilon}} d_M(p_1, p_2) \leq \delta(p_1, p_2) \leq \tilde{a} d_M(p_1, p_2) + \tilde{c} \Rightarrow \\ \xRightarrow{(16)} \quad & \frac{1}{2\hat{\epsilon}A} d_M(\phi_b(p_1), \phi_b(p_2)) - \frac{C}{2\hat{\epsilon}A} \leq \delta(p_1, p_2) = \\ & = \delta(p_1, p_2) \leq \tilde{a} A d_M(\phi_b(p_1), \phi_b(p_2)) + \tilde{a} AC + \tilde{c} \Rightarrow \\ \xRightarrow{(15)} \quad & \frac{1}{2\hat{\epsilon}A\tilde{a}_0} \delta_0(\phi_b(p_1), \phi_b(p_2)) - \frac{\tilde{c}_0}{2\hat{\epsilon}A\tilde{a}_0} - \frac{C}{2\hat{\epsilon}A} \leq \delta(p_1, p_2) = \\ & = \delta(p_1, p_2) \leq \tilde{a}_0 A 2\epsilon_0 \delta_0(\phi_b(p_1), \phi_b(p_2)) \tilde{a} AC + \tilde{c} \end{aligned}$$

which can be rewritten as,

$$\frac{1}{A_{net}} \delta_0(\phi_b(p_1), \phi_b(p_2)) - C_{net} \leq \delta(p_1, p_2) \leq A_{net} \delta_0(\phi_b(p_1), \phi_b(p_2)) + C_{net}$$

where,

$$\begin{aligned} A_{net} &:= A_{net}(m, k_M, \epsilon_0, C, A) := 2A \max\{\hat{\epsilon}\tilde{a}_0, \tilde{a}\epsilon_0\} \geq 1 \\ C_{net} &:= C_{net}(m, k_M, \epsilon_0, C, A) := \max\left\{\tilde{a}AC + \tilde{c}, \frac{1}{2\hat{\epsilon}A\tilde{a}_0}\left(\frac{\tilde{c}_0}{\tilde{a}_0} + C\right)\right\} > 0 \end{aligned}$$

and the Proposition is proved. \square

Proof of Theorem 4.3.

In order to prove the Theorem, by [10] (Lemma 2.5), it suffices to show that an ϵ -net in M is roughly isometric to an ϵ' -net in the product $F_{b_0} \times B$. We remark here that in the proof of [10] (Lemma 5) the maximal property of an ϵ -net is not required, it sufficient that the "net" be a countable, ϵ -separated and ϵ -full set.

We will proceed with the proof by constructing in 2 steps a rough isometry ϕ between countable, separated full sets in M and in $F_{b_0} \times B$.

In *Step 1.* we combine the diffeomorphisms $\varphi_{(\gamma_{[b, b_0]})}$ with two countable maximal separated sets, P_0 in the fiber $F_{b_0} \subset M$ and P_B in B , in a fashion that will produce a suitable countable separated full set P in M . We also show that the product $P_0 \times P_B$ is a countable separated full set in $F_{b_0} \times B$.

Then, in *Step 2.* we introduce a bijection ϕ from P to $P_0 \times P_B$, which will turn out to be the rough isometry between discrete approximations of M and $F_{b_0} \times B$, as mentioned above.

Step 1.

Let the positive constants A, C and α, β , be as in conditions **(RIF)** and **(HLC)**, respectively. Let us choose and fix two constants $\epsilon_0 > 0$ and $\epsilon_B > 0$ satisfying the inequalities (see (4)),

$$\epsilon_0 > C \cdot (1 + A^2) \quad \text{and} \quad \epsilon_B > \left(\frac{\epsilon_0 - C}{A} + \beta\right) \cdot \alpha$$

We first define two countable sets $P_0 \subseteq F_{b_0} \subset M$ and $P_B \subseteq B$, with $b_0 \in P_B$, where P_0 is a maximal ϵ_0 -separated set,

$$\forall p, q \in P_0, p \neq q \Rightarrow d_M(p, q) \geq \epsilon_0$$

and P_B is a maximal ϵ_B -separated set,

$$\forall b_1, b_2 \in P_B, b_1 \neq b_2 \Rightarrow d_B(b_1, b_2) \geq \epsilon_B$$

and then we introduce the net structure $N_0 = \{N_0(p) : p \in P_0\}$ of P_0 given by,

$$N_0(p) = \{q \in P_0 : 0 < d_M(p, q) \leq 2\epsilon_0\}$$

and $N_B = \{N_B(b) : b \in P_B\}$ the net structure of P_B defined by,

$$N_B(b) = \{\hat{b} \in P_B : 0 < d_B(b, \hat{b}) \leq 2\epsilon_B\}$$

Observe that **Proposition 2.3** implies P_0 is ϵ_0 -full in F_{b_0} and P_B is ϵ_B -full in B .

We now, construct P a countable $\left(\frac{\epsilon_0 - C}{A}\right)$ -separated full set in M .

For each $b \in P_B$, let us look first at $\varphi_{(\gamma_{[b, b_0]})}^{-1}(P_0) \subseteq F_b$.

We claim that,

$$\varphi_{(\gamma_{[b, b_0]})}^{-1}(P_0) \text{ is a countable } \left(\frac{\epsilon_0 - C}{A}\right)\text{-separated subset of } F_b \quad (17)$$

The set $\varphi_{(\gamma_{[b, b_0]})}^{-1}(P_0)$ is countable, due to the fact that P_0 is countable and $\varphi_{(\gamma_{[b, b_0]})}$ is bijective.

Notice that by (5), $\left(\frac{\epsilon_0 - C}{A}\right) > 0$

Now, for $b \in P_B$ let us consider either $b = b_0$ or $b \neq b_0$.

If $\underline{b = b_0}$, since P_0 is ϵ_0 -separated, by (6) we have $\frac{\epsilon_0 - C}{A} < \epsilon_0$, so we conclude that $\varphi_{(\gamma_{[b, b_0]})}^{-1}(P_0)$ is $\left(\frac{\epsilon_0 - C}{A}\right)$ -separated, which is claim (17).

If $\underline{b \neq b_0}$, because $\varphi_{(\gamma_{[b, b_0]})}$ is a diffeomorphism, we have,

$$\begin{aligned} \forall p, q \in \varphi_{(\gamma_{[b, b_0]})}^{-1}(P_0) : p \neq q &\Rightarrow \\ \Rightarrow \varphi_{(\gamma_{[b, b_0]})}(p) \neq \varphi_{(\gamma_{[b, b_0]})}(q) \text{ in } P_0 &\Rightarrow \epsilon_0 \leq d_M(\varphi_{(\gamma_{[b, b_0]})}(p), \varphi_{(\gamma_{[b, b_0]})}(q)) \leq \\ \stackrel{\text{(RIF)}}{\leq} Ad_M(p, q) + C &\Rightarrow d_M(p, q) \geq \left(\frac{\epsilon_0 - C}{A}\right) \end{aligned}$$

and claim (17) follows.

Let (see Fig. 8),

$$P := \bigcup_{b \in P_B} \varphi_{(\gamma_{[b, b_0]})}^{-1}(P_0)$$

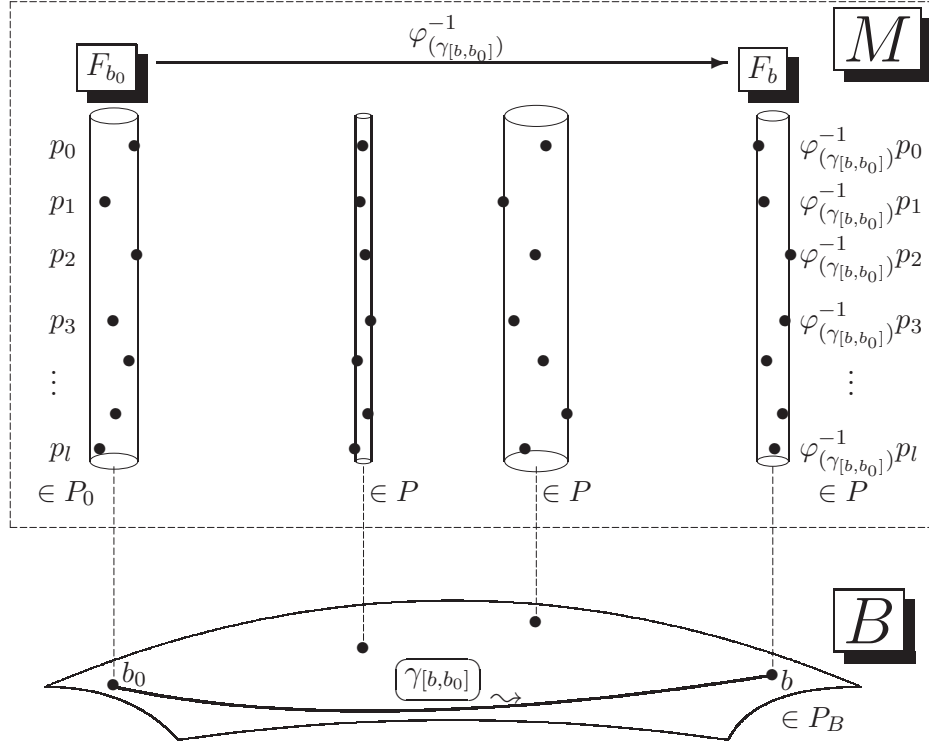


Figure 8: The net $P = \bigcup_{b \in P_B} \varphi_{(\gamma[b,b_0])}^{-1}(P_0)$.

Since P_0, P_B are countable sets and $\varphi_{(\gamma[b,b_0])}$ is a bijection for all $b \in B$, the set P is also countable.

To show that P is $\left(\frac{\epsilon_0 - C}{A}\right)$ -separated, we proceed as follows.

For any $p, q \in P$ such that $p \neq q$, we have only two cases,

(CASE I:) $\exists b \in P_B : p, q \in \varphi_{(\gamma[b,b_0])}^{-1}(P_0)$.

In that case (17) gives us, $d_M(p, q) \geq \frac{\epsilon_0 - C}{A}$.

(CASE II:) $\exists b \in P_B : p \in \varphi_{(\gamma[b,b_0])}^{-1}(P_0)$ and $\exists \tilde{b} \in P_B : q \in \varphi_{(\gamma[\tilde{b},b_0])}^{-1}(P_0)$, where $\pi p = b \neq \tilde{b} = \pi \tilde{b}$.

We claim,

$$d_M(p, q) \geq \frac{1}{\alpha} \epsilon_B - \beta \stackrel{(4)}{>} \frac{\epsilon_0 - C}{A}$$

We will only verify the first inequality, since the second one is the require-

ment (4) on ϵ_B .

Let ς be a general curve parametrized proportionally to .a.l. curve joining p and q in M , with length $\ell(\varsigma)$. In this case $\pi \circ \varsigma$ is a curve joining b and \tilde{b} . In addition, we will denote by $\gamma_{b\tilde{b}}$ the minimal geodesic joining b and \tilde{b} .

For all $x \in M$, we can write $v = v_V \oplus v_H \in T_x M = (VT)_x \oplus (HT)_x$ in a unique way, where $v_V \in (\ker(\pi_*)_x)$ and $v_H \in (\ker(\pi_*)_x)^\perp$. Furthermore, $\|v\|_M = \|v_V \oplus v_H\|_M = \sqrt{\|v_V\|_M^2 + \|v_H\|_M^2}$.

By property **(HLC)**, the facts that $\gamma_{b\tilde{b}}$ is a minimal geodesic and B is complete,

$$\begin{aligned} \ell(\varsigma) &= \int_0^1 \|\varsigma'(t)\|_M dt = \int_0^1 \|\varsigma'_H(t) \oplus \varsigma'_V(t)\|_M dt \geq \int_0^1 \|\varsigma'_H(t)\|_M dt \stackrel{(\text{HLC})}{\geq} \\ &\geq \frac{1}{\alpha} \int_0^1 \|(\pi \circ \varsigma)'(t)\|_B dt - \beta = \frac{1}{\alpha} \ell(\pi \circ \varsigma) - \beta \stackrel{\text{min.geod.}}{\geq} \\ &\geq \frac{1}{\alpha} \ell(\gamma_{b\tilde{b}}) - \beta \stackrel{B\text{complete}}{=} \frac{1}{\alpha} d_B(b, \tilde{b}) - \beta, \quad \forall \varsigma \end{aligned}$$

which is a lower bound on the length of any curve ς joining p and q in M , independent of the curve ς .

Finally, by definition of infimum and from $b, \tilde{b} \in P_B$,

$$d_M(p, q) := \inf_{\varsigma \subset M} \ell(\varsigma) \geq \frac{1}{\alpha} d_B(b, \tilde{b}) - \beta \stackrel{b, \tilde{b} \in P_B}{\geq} \frac{1}{\alpha} \epsilon_B - \beta \stackrel{(9)}{>} 0$$

and the claim is proved.

So, P is $\left(\frac{\epsilon_0 - C}{A}\right)$ -separated.

We introduce the net structure $N_P = \{N_P(p) : p \in P\}$ of P given by,

$$N_P(p) = \left\{ q \in P : 0 < d_M(p, q) \leq 2 \left(\frac{\epsilon_0 - C}{A} \right) \right\}$$

Next, we prove that that P is $[(\epsilon_0 + C)A + \alpha\epsilon_B + \beta]$ -full in M , i.e.

$$M = B_{[(\epsilon_0 + C)A + \alpha\epsilon_B + \beta]} P = \{x \in M : d_M(x, P) < (\epsilon_0 + C)A + \alpha\epsilon_B + \beta\}$$

We want to show that,

$$d_M(x, P) := \inf_{p \in P} d_M(x, p) < (\epsilon_0 + C)A + \alpha\epsilon_B + \beta, \quad \forall x \in M$$

Let $x \in M$. Either $x \in P$ or $x \in M \setminus P$.

If $x \in P$, then $d_M(x, P) = 0 < (\epsilon_0 + C)A + \alpha\epsilon_B + \beta$.

If $x \in M \setminus P$, let $b := \pi x \in B$.

There are two cases for such b , either $b \in P_B$ or $b \in B \setminus P_B$.

(CASE I:) If $b \in P_B$, since x is not in P , P_0 is maximal, property **(RIF)** holds, and $b \in P_B$ implies $\varphi_{(\gamma_{[b, b_0]})}^{-1}(p_0) \in P, \forall p_0 \in P_0$, then,

$$\begin{aligned}
b \in P_B & \xrightarrow{x \in M \setminus P} \varphi_{(\gamma_{[b, b_0]})} x \in F_{b_0} \setminus P_0 \xrightarrow{P_0 \text{ max.}} \exists p_0 \in P_0 : d_M(\varphi_{(\gamma_{[b, b_0]})} x, p_0) < \epsilon_0 \\
& \xrightarrow{\text{(RIF)}} \frac{1}{A} d_M(x, \varphi_{(\gamma_{[b, b_0]})}^{-1}(p_0)) - C \leq d_M(\varphi_{(\gamma_{[b, b_0]})} x, p_0) < \epsilon_0 \Rightarrow \\
& \Rightarrow d_M(x, \underbrace{\varphi_{(\gamma_{[b, b_0]})}^{-1}(p_0)}_{\in P}) < (\epsilon_0 + C)A < (\epsilon_0 + C)A + \underbrace{\alpha\epsilon_B + \beta}_{>0} \Rightarrow \\
& \Rightarrow \inf_{p \in P} d_M(x, p) \leq d_M(x, \varphi_{(\gamma_{[b, b_0]})}^{-1}(p_0)) < (\epsilon_0 + C)A + \alpha\epsilon_B + \beta \\
& \Rightarrow d_M(x, P) < (\epsilon_0 + C)A + \alpha\epsilon_B + \beta
\end{aligned}$$

(CASE II:) If $b \in B \setminus P_B$, by the maximality of P_B there exists $\bar{b} \in P_B : d_B(b, \bar{b}) < \epsilon_B$.

We wish to obtain $\bar{p} \in P$ satisfying $d_M(x, \bar{p}) \leq (\epsilon_0 + C)A + \alpha\epsilon_B + \beta$, which will be accomplished as follows.

Let,

$$\gamma_{b\bar{b}} : [t_1, t_2] \longrightarrow B, \gamma_{b\bar{b}}(t_1) := b, \gamma_{b\bar{b}}(t_2) := \bar{b}$$

be a parametrization proportional to arclength of a minimal geodesic joining b and \bar{b} in B , and let $\Gamma_{b\bar{b}}$ be its unique horizontal lift through x , which in particular satisfies

$$\Gamma_{b\bar{b}}(t_2) \in F_{\bar{b}} \quad (18)$$

We have, by (18), the fact that P_0 is ϵ_0 -full in F_{b_0} and the definition of infimum,

$$\begin{aligned}
& \Gamma_{b\bar{b}}(t_2) \in F_{\bar{b}} \Rightarrow \varphi_{(\gamma_{[\bar{b}, b_0]})}(\Gamma_{b\bar{b}}(t_2)) \in F_{b_0} \Rightarrow \\
& \xrightarrow{P_0 \text{ full}} d_M\left(\varphi_{(\gamma_{[\bar{b}, b_0]})}(\Gamma_{b\bar{b}}(t_2)), P_0\right) < \epsilon_0 \Rightarrow \\
& \xrightarrow{\text{infimum}} \exists p_0 \in P_0 : d_M\left(\varphi_{(\gamma_{[\bar{b}, b_0]})}(\Gamma_{b\bar{b}}(t_2)), p_0\right) < \epsilon_0 \quad (19)
\end{aligned}$$

We claim that the desired $\bar{p} \in P$ is exactly $\varphi_{(\gamma_{[\bar{b}, b_0]})}^{-1}(p_0)$.

Indeed,

$$\bar{b} \in P_B, \quad \varphi_{(\gamma_{[\bar{b}, b_0]})}^{-1}(p_0) \in \varphi_{(\gamma_{[\bar{b}, b_0]})}^{-1}(P_0) \Rightarrow \varphi_{(\gamma_{[\bar{b}, b_0]})}^{-1}(p_0) \in P$$

Furthermore, by the triangle inequality,

$$d_M \left(\varphi_{(\gamma_{[\bar{b}, b_0]})}^{-1}(p_0), x \right) \leq d_M \left(\varphi_{(\gamma_{[\bar{b}, b_0]})}^{-1}(p_0), \Gamma_{b\bar{b}}(t_2) \right) + d_M \left(\Gamma_{b\bar{b}}(t_2), x \right) \quad (20)$$

In addition we have,

$$\begin{aligned} & \frac{1}{A} d_M \left(\Gamma_{b\bar{b}}(t_2), \varphi_{(\gamma_{[\bar{b}, b_0]})}^{-1}(p_0) \right) - C \stackrel{(\mathbf{RIF})}{\leq} d_M \left(\varphi_{(\gamma_{[\bar{b}, b_0]})}(\Gamma_{b\bar{b}}(t_2)), p_0 \right) \Rightarrow \\ \Rightarrow & d_M \left(\Gamma_{b\bar{b}}(t_2), \varphi_{(\gamma_{[\bar{b}, b_0]})}^{-1}(p_0) \right) \leq \left[d_M \left(\varphi_{(\gamma_{[\bar{b}, b_0]})}(\Gamma_{b\bar{b}}(t_2)), p_0 \right) + C \right] A < \\ & \stackrel{(19)}{<} (\epsilon_0 + C)A \end{aligned} \quad (21)$$

Also,

$$\begin{aligned} d_M \left(\Gamma_{b\bar{b}}(t_2), x \right) &= d_M \left(\Gamma_{b\bar{b}}(t_2), \Gamma_{b\bar{b}}(t_1) \right) \stackrel{\text{infimum}}{\leq} \ell(\Gamma_{b\bar{b}}) = \int_0^1 \|\Gamma'_{b\bar{b}}(t)\|_M dt \\ &\stackrel{(\mathbf{HLC})}{\leq} \alpha \int_0^1 \|(\pi \circ \Gamma_{b\bar{b}})'(t)\|_B dt + \beta \stackrel{(\mathbf{hl.1})}{=} \alpha \int_0^1 \|\gamma'_{b\bar{b}}(t)\|_B dt + \beta = \\ &= \alpha \ell(\gamma_{b\bar{b}}) + \beta \stackrel{\text{min.geod.}}{=} \alpha d_B(b, \bar{b}) + \beta \stackrel{\text{def.}\bar{b}}{<} \alpha \epsilon_B + \beta \end{aligned} \quad (22)$$

Thus, by combining (20), (21) and (22),

$$d_M \left(\varphi_{(\gamma_{[\bar{b}, b_0]})}^{-1}(p_0), x \right) \leq (\epsilon_0 + C)A + \alpha \epsilon_B + \beta$$

which in turn implies,

$$d_M(x, P) = \inf_{p \in P} d_M(x, p) \leq d_M(x, \underbrace{\varphi_{(\gamma_{[\bar{b}, b_0]})}^{-1}(p_0)}_{\in P}) \leq (\epsilon_0 + C)A + \alpha \epsilon_B + \beta$$

and we conclude that P is $[(\epsilon_0 + C)A + \alpha \epsilon_B + \beta]$ -full in M .

In what follows, we show that $P_0 \times P_B$ is countable, $(\epsilon_0 + \epsilon_B)$ -separated, and $(\epsilon_0 + \epsilon_B)$ -full in $F_{b_0} \times B$.

In $F_{b_0} \times B$ we have the induced product metric from $M \times B$,

$$d_{\times} \left((x, b), (\tilde{x}, \tilde{b}) \right) := d_M(x, \tilde{x}) + d_B(b, \tilde{b})$$

for all $x, \tilde{x} \in F_{b_0}$ and $b, \tilde{b} \in B$.

$P_0 \times P_B$ being countable comes from the fact that both P_0 and P_B have that property.

Let $(x, b) \neq (\tilde{x}, \tilde{b}) \in P_0 \times P_B$. Since P_0 is ϵ_0 -separated and P_B is ϵ_B -separated,

$$d_{\times} \left((x, b), (\tilde{x}, \tilde{b}) \right) = d_M(x, \tilde{x}) + d_B(b, \tilde{b}) \geq \epsilon_0 + \epsilon_B$$

and $P_0 \times P_B$ is $(\epsilon_0 + \epsilon_B)$ -separated.

To prove that $P_0 \times P_B$ is $(\epsilon_0 + \epsilon_B)$ -full in $F_{b_0} \times B$, i.e.,

$$F_{b_0} \times B = \{(x, b) \in F_{b_0} \times B : d_{\times}((x, b), P_0 \times P_B) < \epsilon_0 + \epsilon_B\}$$

we need to show that,

$$d_{\times}((x, b), P_0 \times P_B) < \epsilon_0 + \epsilon_B, \quad \forall (x, b) \in F_{b_0} \times B$$

Let $(x, b) \in F_{b_0} \times B$.

Since, P_0 is ϵ_0 -full in F_{b_0} , there exists $p_0 \in P_0 : d_M(p_0, x) < \epsilon_0$. Similarly, P_B being ϵ_B -full in B , implies that there exists $\bar{b} \in B : d_B(\bar{b}, b) < \epsilon_B$.

Therefore,

$$\begin{aligned} d_{\times}((x, b), P_0 \times P_B) &\stackrel{\text{inf.}}{\leq} d_{\times}((x, b), (p_0, \bar{b})) = d_M(x, p_0) + d_B(b, \bar{b}) < \\ &< \epsilon_0 + \epsilon_B \end{aligned}$$

and since (x, b) is arbitrary, we conclude that $P_0 \times P_B$ is $(\epsilon_0 + \epsilon_B)$ -full in $F_{b_0} \times B$.

Step 2.

Let us initially define some notation as well as provide a geometric interpretation of a "net". We will assume that all nets are connected, since we can repeat the argument on each connected component of the underlying manifold.

Graphically, we will describe an element of an ϵ -net as the center of a ball of radius $\frac{\epsilon}{2}$, which can be visualized as a coin with diameter ϵ . So, the control of distances between elements in an ϵ -net allows us to describe it as a countable set of coins, which can touch boundaries but can never overlap. We will call such element an ϵ -**coin** (see Fig. 9).

We define a map ϕ from $P \subset M$ into $P_0 \times P_B \subset F_{b_0} \times B$ as follows,

$$\begin{aligned} \phi : P = \bigcup_{b \in P_B} \varphi_{(\gamma_{[b, b_0]})}^{-1}(P_0) &\longrightarrow P_0 \times P_B \\ p &\longmapsto \phi p := \left(\varphi_{(\gamma_{[\pi p, b_0]})} p, \pi p \right) \end{aligned}$$

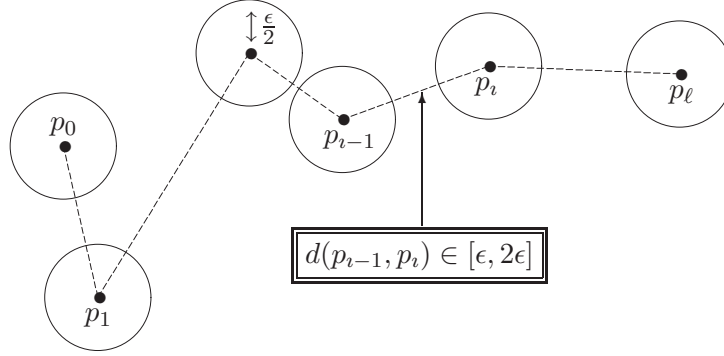


Figure 9: A discrete path $(p_0, p_1, \dots, p_\ell)$ in an ϵ -net, and its elements regarded as centers of coins with diameter ϵ .

(Claim 1) ϕ is well-defined.

If $p \in \varphi_{(\gamma_{b_0})}^{-1}(P_0) = P_0$, then $\pi p = b_0$ and $\phi p = (\varphi_{(\gamma_{b_0})} p, b_0) = (p, b_0) \in P_0 \times P_B$.

If $p \in \varphi_{(\gamma_{[b, b_0]})}^{-1}(P_0)$, where $b \in P_B, b \neq b_0$, then $\pi p = b$ and $\phi p = (\varphi_{(\gamma_{[b, b_0]})} p, b) \in P_0 \times P_B$.

(Claim 2) ϕ is 1-1.

Let $p, q \in P$ and assume that $\phi p = \phi q$.

Thus,

$$\phi p = \phi q \Rightarrow \begin{cases} \varphi_{(\gamma_{[\pi p, b_0]})} p = \varphi_{(\gamma_{[\pi q, b_0]})} q \\ \pi p = \pi q =: b \end{cases} \Rightarrow \varphi_{(\gamma_{[b, b_0]})} p = \varphi_{(\gamma_{[b, b_0]})} q \stackrel{\text{diff eo.}}{\Rightarrow} p = q$$

and ϕ is injective.

(Claim 3) ϕ is onto.

Let $(p_0, b) \in P_0 \times P_B$ and define $p \in F_b$ by,

$$p := \varphi_{(\gamma_{[b, b_0]})}^{-1} p_0 \in \varphi_{(\gamma_{[b, b_0]})}^{-1}(P_0) \subseteq F_b$$

where, $\pi p = b$ and $p = p_0$ if $b = b_0$.

Thus,

$$\phi p = (\varphi_{(\gamma_{[\pi p, b_0]})} p, \pi p) = (\varphi_{(\gamma_{[b, b_0]})} p, b) = (p_0, b)$$

and ϕ is surjective.

(Claim 4) ϕ satisfies **(RI.1)**.

By (Claim 3), for any given $\epsilon > 0$, we have,

$$P_0 \times P_B = \phi(P) = \mathcal{B}_\epsilon \phi(P) = \{(p_0, b) \in P_0 \times P_B : d_\times((p_0, b), \phi(P)) < \epsilon\}$$

in other words, ϕ is ϵ -full in $P_0 \times P_B$ for any $\epsilon > 0$, which is exactly **(RI.1)** for ϕ .

(Claim 5) ϕ satisfies **(RI.2)**.

We want to show that there exist constants $a \geq 1$ and $c > 0$ satisfying,

$$\frac{1}{a} \delta_P(p, q) - c \leq \delta_\times(\phi p, \phi q) \leq a \delta_P(p, q) + c, \quad \forall p, q \in P$$

where δ_P is the combinatorial metric of P , and δ_\times is the discrete product metric of $P_0 \times P_B$ given by,

$$\delta_\times((p, b), (\tilde{p}, \tilde{b})) := \delta_{P_0}(p, \tilde{p}) + \delta_{P_B}(b, \tilde{b})$$

for all $(p, b), (\tilde{p}, \tilde{b}) \in P_0 \times P_B$.

In terms of δ_{P_0} and δ_{P_B} , the condition we want to verify for ϕ , translates into, $\exists a \geq 1, \exists c > 0 : \forall p, q \in P$,

$$\frac{1}{a} \delta_P(p, q) - c \leq \delta_{P_0} \left(\varphi_{(\gamma_{[\pi p, b_0]})} p, \varphi_{(\gamma_{[\pi q, b_0]})} q \right) + \delta_{P_B}(\pi p, \pi q) \leq a \delta_P(p, q) + c \quad (23)$$

Indeed, let $p, q \in P = \bigcup_{b \in P_B} \varphi_{(\gamma_{[b, b_0]})}^{-1}(P_0)$.

Let γ_{\min} be a minimal geodesic joining π_q to π_p in B , and let its unique horizontal lift through q be parametrized by $\Gamma_q : [t_1, t_2] \rightarrow M$.

By the triangle inequality, the definition of distance, **Proposition 3.8**, trivial holonomy **(TH)**, and property **(RIF)**, we have (see Fig. 10)

$$\begin{aligned} d_M(p, q) &\stackrel{\Delta}{\leq} d_M(p, \varphi_{(\gamma_{\min})} q) + d_M(\varphi_{(\gamma_{\min})} q, q) \leq \\ &\stackrel{dist.}{\leq} d_M(p, \varphi_{(\gamma_{\min})} q) + \ell(\Gamma_q) \leq \\ &\stackrel{(RIF)}{\leq} Ad_M(\varphi_{(\gamma_{[\pi p, b_0]})} p, \varphi_{(\gamma_{[\pi p, b_0]})} \varphi_{(\gamma_{\min})} q) + AC + \ell(\Gamma_q) \leq \\ &\stackrel{Prop 3.8}{\leq} Ad_M(\varphi_{(\gamma_{[\pi p, b_0]})} p, \varphi_{(\gamma_{[\pi p, b_0]})} \varphi_{(\gamma_{\min})} q) + AC + \alpha [\ell(\gamma_{\min}) + \beta(t_2 - t_1)] \\ &\stackrel{(TH)}{\leq} Ad_M(\varphi_{(\gamma_{[\pi p, b_0]})} p, \varphi_{(\gamma_{[\pi q, b_0]})} q) + AC + \alpha [\ell(\gamma_{\min}) + \beta(t_2 - t_1)] \\ &\stackrel{dist.}{=} \alpha d_B(\pi p, \pi q) + Ad_M(\varphi_{(\gamma_{[\pi p, b_0]})} p, \varphi_{(\gamma_{[\pi q, b_0]})} q) + \alpha \beta(t_2 - t_1) + AC \end{aligned}$$

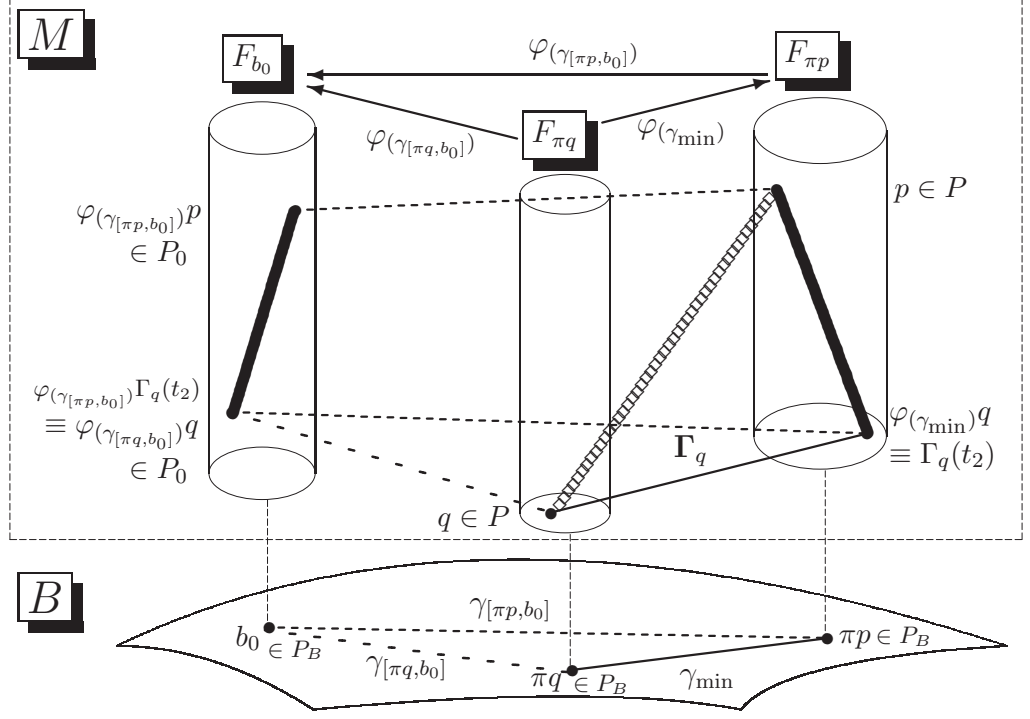


Figure 10: An upper bound for the distance $d_M(p, q)$, via the triangle inequality.

i.e.,

$$d_M(p, q) \leq \alpha d_B(\pi p, \pi q) + A d_M(\varphi(\gamma_{[\pi p, b_0]})p, \varphi(\gamma_{[\pi q, b_0]})q) + AC + \alpha\beta(t_2 - t_1) \quad (24)$$

Now, by Lemma 2.5 in [10] we have for the nets P , P_0 and P_B , respectively,

$$\exists \hat{a}(m, k_M, \hat{\epsilon}) \geq 1, \exists \hat{c}(m, k_M, \hat{\epsilon}) > 0 : \frac{1}{\hat{a}} \delta_P(p, q) - \frac{\hat{c}}{\hat{a}} \leq d_M(p, q) \quad (25)$$

$$d_M(\varphi(\gamma_{[\pi p, b_0]})p, \varphi(\gamma_{[\pi q, b_0]})q) \leq 2\epsilon_0 \delta_{P_0}(\varphi(\gamma_{[\pi p, b_0]})p, \varphi(\gamma_{[\pi q, b_0]})q) \quad (26)$$

$$d_B(\pi p, \pi q) \leq 2\epsilon_B \delta_{P_B}(\pi p, \pi q) \quad (27)$$

where $m = \dim M$, $k_M > 0$ [see 2.4(BG.R)] and $\hat{\epsilon} > 0$ [see 3.8] are constants.

If we combine (25), (26), and (27) into (24), we obtain,

$$\frac{1}{\hat{a}} \delta_P(p, q) - \frac{\hat{c}}{\hat{a}} \stackrel{(25)}{\leq} d_M(p, q) \leq$$

$$\begin{aligned}
 & \stackrel{(24)}{\leq} \alpha d_B(\pi p, \pi q) + Ad_M(\varphi(\gamma_{[\pi p, b_0]})p, \varphi(\gamma_{[\pi q, b_0]})q) + AC + \alpha\beta(t_2 - t_1) \\
 & \stackrel{(26)}{\leq} \alpha d_B(\pi p, \pi q) + A \left(2\epsilon_0 \delta_{P_0}(\varphi(\gamma_{[\pi p, b_0]})p, \varphi(\gamma_{[\pi q, b_0]})q) \right) + AC + \\
 & \quad + \alpha\beta(t_2 - t_1) \leq \\
 & \stackrel{(27)}{\leq} \alpha 2\epsilon_B \delta_{P_B}(\pi p, \pi q) + A \left(2\epsilon_0 \delta_{P_0}(\varphi(\gamma_{[\pi p, b_0]})p, \varphi(\gamma_{[\pi q, b_0]})q) \right) + AC + \\
 & \quad + \alpha\beta(t_2 - t_1) \Rightarrow \\
 & \Rightarrow \frac{1}{\hat{a}} \delta_P(p, q) - \frac{\hat{c}}{\hat{a}} \leq 2A\epsilon_0 \delta_{P_0}(\varphi(\gamma_{[\pi p, b_0]})p, \varphi(\gamma_{[\pi q, b_0]})q) + 2\alpha\epsilon_B \delta_{P_B}(\pi p, \pi q) + \\
 & \quad + AC + \alpha\beta(t_2 - t_1) \Rightarrow \\
 & \Rightarrow \frac{1}{\hat{a}} \delta_P(p, q) - \frac{\hat{c}}{\hat{a}} \leq \max \{2A\epsilon_0, 2\alpha\epsilon_B\} \cdot \delta_{\times}(\phi p, \phi q) + AC + \alpha\beta(t_2 - t_1) \\
 & \Rightarrow \frac{1}{\hat{a} \cdot \max \{2A\epsilon_0, 2\alpha\epsilon_B\}} \delta_P(p, q) - \\
 & \quad - \left[\frac{\hat{c}}{\hat{a}} + AC + \alpha\beta(t_2 - t_1) \right] \frac{1}{\max \{2A\epsilon_0, 2\alpha\epsilon_B\}} \leq \delta_{\times}(\phi p, \phi q) \quad (28)
 \end{aligned}$$

In what follows, we will produce the inequality that completes (28) into the searched condition (23).

Let us denote $l := \delta_P(p, q)$.

Define (y_0, y_1, \dots, y_l) a discrete path in P of minimum length l joining p to q . Hence, (y_0, y_1, \dots, y_l) has the following properties,

$$\begin{aligned}
 & y_0 := p \in P, \quad y_l := q \in P, \\
 & d_M(y_i, y_j) \geq \hat{c}, \quad i, j = 0, 1, \dots, l \quad (i \neq j) \\
 & d_M(y_{i-1}, y_i) \leq 2\hat{c}, \quad i = 1, \dots, l \\
 & \Rightarrow \delta_P(y_{i-1}, y_i) = 1, \quad i = 1, \dots, l \\
 & \stackrel{\text{def } P}{\Rightarrow} \pi y_i \in P_B, \quad i = 0, 1, \dots, l \\
 & \stackrel{\text{def } P}{\Rightarrow} \varphi(\gamma_{[\pi y_i, b_0]})y_i \in P_0, \quad i = 0, 1, \dots, l \\
 & \stackrel{\text{def } P_B}{\Rightarrow} d_B(\pi y_i, \pi y_j) \geq \epsilon_B, \quad i, j = 0, 1, \dots, l \quad (i \neq j) \quad (29)
 \end{aligned}$$

Next, we will compare l with $\delta_{P_B}(\pi p, \pi q)$.

Notice that because we are assuming **(HLC)**, by **Lemma 3.4** we obtain for any $x, y \in M$,

$$d_M(x, y) \geq \frac{1}{\alpha} d_B(\pi x, \pi y) - \beta \quad (30)$$

So with (29), (30) and (12), we get,

$$\begin{aligned}
&\Rightarrow 2\hat{\epsilon} \stackrel{(29)}{\geq} d_M(y_{l-1}, y_l) \stackrel{(30)}{\geq} \frac{1}{\alpha} d_B(\pi y_{l-1}, \pi y_l) - \beta \Rightarrow \\
&\Rightarrow \alpha(2\hat{\epsilon} + \beta) \geq d_B(\pi y_l, \pi y_{l-1}) \Rightarrow \\
&\Rightarrow 2\epsilon_B \stackrel{(12)}{>} \alpha(2\hat{\epsilon} + \beta) \geq d_B(\pi y_l, \pi y_{l-1}) \stackrel{(29)}{\geq} \epsilon_B \Rightarrow \\
&\Rightarrow 2\epsilon_B \geq d_B(\pi y_l, \pi y_{l-1}) \geq \epsilon_B, \quad \forall l = 1, \dots, l \quad (31)
\end{aligned}$$

Since (31) holds, we obtain a discrete path $(\pi y_0, \pi y_1, \dots, \pi y_{l-1}, \pi y_l)$ in P_B , connecting πp to πq .

Therefore, by the definition of δ_{P_B} , we conclude that,

$$\delta_{P_B}(\pi p, \pi q) \leq l = \delta_P(p, q) \quad (32)$$

Now, we will compare l with $\delta_{P_0}(\varphi(\gamma_{[\pi p, b_0]})p, \varphi(\gamma_{[\pi q, b_0]})q)$.

By Lemma 2.5 [10], we have for the nets P_0 , P and P_B , respectively,

$$\begin{aligned}
&\exists a_0(m, k_M, \epsilon_0) \geq 1, \exists c_0(m, k_M, \epsilon_0) > 0 : \\
&\delta_{P_0}(p_1, p_2) \leq a_0 \cdot d_M(p_1, p_2) + c_0, \quad \forall p_1, p_2 \in P_0 \quad (33)
\end{aligned}$$

$$d_M(p_3, p_4) \leq 2\hat{\epsilon} \delta_P(p_3, p_4), \quad \forall p_3, p_4 \in P \quad (34)$$

$$d_B(\pi y_{l-1}, \pi y_l) \leq 2\epsilon_B \delta_{P_B}(\pi y_{l-1}, \pi y_l), \quad \forall l = 1, \dots, l \quad (35)$$

Initially, for each $i = 1, \dots, l$, let us look at

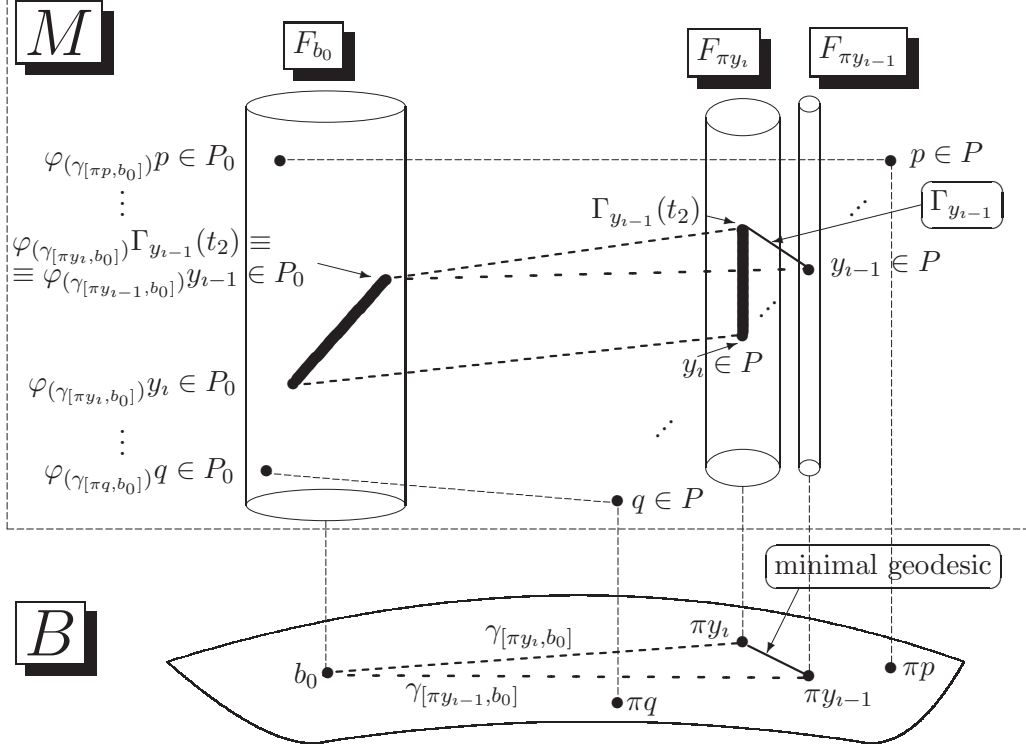
$$\delta_{P_0}(\varphi(\gamma_{[\pi y_{i-1}, b_0]})y_{i-1}, \varphi(\gamma_{[\pi y_i, b_0]})y_i)$$

which is well defined because of properties (29).

For each unique (see remark in the beginning of *Step 1*.) minimal geodesic in B joining πy_{i-1} to πy_i , let its unique horizontal lift through y_{i-1} be denoted by $\Gamma_{y_{i-1}} : [t_1, t_2] \longrightarrow M$.

By trivial holonomy **(TH)**, (33) and **(RIF)**, we may write (see Fig. 11)

$$\begin{aligned}
&\delta_{P_0}(\varphi(\gamma_{[\pi y_{i-1}, b_0]})y_{i-1}, \varphi(\gamma_{[\pi y_i, b_0]})y_i) \leq \\
&\stackrel{(\mathbf{TH})}{\leq} \delta_{P_0}(\varphi(\gamma_{[\pi y_i, b_0]})\Gamma_{y_{i-1}}(t_2), \varphi(\gamma_{[\pi y_i, b_0]})y_i) \leq \\
&\stackrel{(33)}{\leq} a_0 d_M(\varphi(\gamma_{[\pi y_i, b_0]})\Gamma_{y_{i-1}}(t_2), \varphi(\gamma_{[\pi y_i, b_0]})y_i) + c_0 \leq \\
&\stackrel{(\mathbf{RIF})}{\leq} a_0 [Ad_M(\Gamma_{y_{i-1}}(t_2), y_i) + C] + c_0 \quad (36)
\end{aligned}$$

Figure 11: y_{i-1} , the lift $\Gamma_{y_{i-1}}(t_2)$, and their counterparts in the fiber F_{b_0} .

Now, that (34), **Proposition 3.8** and (35) hold, implies

$$\begin{aligned}
 d_M(\Gamma_{y_{i-1}}(t_2), y_i) &\stackrel{\Delta}{\leq} d_M(\Gamma_{y_{i-1}}(t_2), y_{i-1}) + d_M(y_{i-1}, y_i) \leq \\
 &\stackrel{(34)}{\leq} d_M(\Gamma_{y_{i-1}}(t_2), y_{i-1}) + 2\hat{\epsilon} \overbrace{\delta_P(y_{i-1}, y_i)}^{(31)_1} \stackrel{dist.}{\leq} \ell(\Gamma_{y_{i-1}}) + 2\hat{\epsilon} \leq \\
 &\stackrel{Prop 3.8}{\leq} \alpha [\ell(\gamma_{\pi y_{i-1} \pi y_i}) + \beta(t_2 - t_1)] + 2\hat{\epsilon} = \\
 &\stackrel{dist.}{=} \alpha d_B(\pi y_{i-1}, \pi y_i) + \alpha \beta(t_2 - t_1) + 2\hat{\epsilon} \leq \\
 &\stackrel{(35)}{\leq} \alpha [2\epsilon_B \underbrace{\delta_{P_B}(\pi y_{i-1}, \pi y_i)}_{(31)_1}] + \alpha \beta(t_2 - t_1) + 2\hat{\epsilon} = \\
 &\stackrel{(31)}{=} \alpha 2\epsilon_B + \alpha \beta(t_2 - t_1) + 2\hat{\epsilon}
 \end{aligned} \tag{37}$$

By combining and (36) and (37), we get

$$\begin{aligned}
& \delta_{P_0} \left(\varphi_{(\gamma_{[\pi y_{i-1}, b_0]})} y_{i-1}, \varphi_{(\gamma_{[\pi y_i, b_0]})} y_i \right) \leq \\
& \stackrel{(36)}{\leq} a_0 [Ad_M(\Gamma_{y_{i-1}}(t_2), y_i) + C] + c_0 \leq \\
& \stackrel{(37)}{\leq} a_0 \{A[2\alpha\epsilon_B + \alpha\beta(t_2 - t_1) + 2\hat{\epsilon}] + C\} + c_0 = \\
& = a_0 A[2\alpha\epsilon_B + \alpha\beta(t_2 - t_1) + 2\hat{\epsilon}] + a_0 C + c_0
\end{aligned} \tag{38}$$

If we sum (38) over $i = 1, \dots, l$, we may write

$$\begin{aligned}
& \delta_{P_0} \left(\varphi_{(\gamma_{[\pi p, b_0]})} p, \varphi_{(\gamma_{[\pi q, b_0]})} q \right) \leq \\
& \stackrel{\triangle}{\leq} \sum_{i=1}^l \delta_{P_0} \left(\varphi_{(\gamma_{[\pi y_{i-1}, b_0]})} y_{i-1}, \varphi_{(\gamma_{[\pi y_i, b_0]})} y_i \right) \leq \\
& \stackrel{(38)}{\leq} l \{a_0 A[2\alpha\epsilon_B + \alpha\beta(t_2 - t_1) + 2\hat{\epsilon}] + a_0 C + c_0\} = \\
& = \delta_P(p, q) \{a_0 A[2\alpha\epsilon_B + \alpha\beta(t_2 - t_1) + 2\hat{\epsilon}] + a_0 C + c_0\}
\end{aligned} \tag{39}$$

Next, by using inequalities (32) and (39),

$$\begin{aligned}
& \delta_{\times}(\phi p, \phi q) = \delta_{P_0} \left(\varphi_{(\gamma_{[\pi p, b_0]})} p, \varphi_{(\gamma_{[\pi q, b_0]})} q \right) + \delta_{P_B}(\pi p, \pi q) \stackrel{(32)}{\leq} \\
& \stackrel{(39)}{\leq} \{a_0 A[2\alpha\epsilon_B + \alpha\beta(t_2 - t_1) + 2\hat{\epsilon}] + a_0 C + c_0\} \delta_P(p, q) + \delta_P(p, q) = \\
& = \{a_0 A[2\alpha\epsilon_B + \alpha\beta(t_2 - t_1) + 2\hat{\epsilon}] + a_0 C + c_0 + 1\} \delta_P(p, q)
\end{aligned} \tag{40}$$

Finally, we combine inequalities (28) and (40), and obtain

$$\begin{aligned}
& \frac{1}{\hat{a} \cdot \max\{2A\epsilon_0, 2\alpha\epsilon_B\}} \delta_P(p, q) - \\
& - \left[\frac{\hat{c}}{\hat{a}} + \alpha\beta(t_2 - t_1) + AC \right] \frac{1}{\max\{2A\epsilon_0, 2\alpha\epsilon_B\}} \stackrel{(28)}{\leq} \delta_{\times}(\phi p, \phi q) \leq \\
& \stackrel{(40)}{\leq} \{a_0 A[2\alpha\epsilon_B + \alpha\beta(t_2 - t_1) + 2\hat{\epsilon}] + a_0 C + c_0 + 1\} \delta_P(p, q)
\end{aligned}$$

which is the required property (23)

$$\frac{1}{a} \delta_P(p, q) - c \leq \delta_{\times}(\phi p, \phi q) \leq a \delta_P(p, q) + c$$

with the universal constants given by

$$\begin{aligned} a &:= \max \{a_0 A [2\alpha\epsilon_B + \alpha\beta(t_2 - t_1) + 2\hat{\epsilon}] + a_0 C + c_0 + 1, \\ &\quad \hat{a} \cdot \max \{2A\epsilon_0, 2\alpha\epsilon_B\}\} \geq 1 \\ c &:= \left[\frac{\hat{c}}{\hat{a}} + \alpha\beta(t_2 - t_1) + AC \right] \frac{1}{\max \{2A\epsilon_0, 2\alpha\epsilon_B\}} > 0 \end{aligned}$$

and so ϕ satisfies **(RI.2)**.

This concludes the proof of the Theorem.

□

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